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Optimization of Two-Dimensional Scramjet Inlets

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The classical optimization of two-dimensional supersonic inlets for maximum total pressure recovery is extended to two-dimensional scramjet inlets. The result that optimal supersonic inlets have shock waves of equal strength is often applied to scramjets. However the typical flow turning constraint required for scramjets, along with the lack of a terminating normal shock, lead to a more involved optimization problem. Despite this, optimization by the method of Lagrange multipliers indicates that scramjet inlets with maximum total pressure recovery have external shocks with almost equal strength. The optimal total pressure recovery and turning angles for some typical scramjet inlet configurations with up to five shocks are presented.

Nomenclature

F = $G + \Sigma_{i=1}^{m} \lambda_i \psi_i$
f = $\{[(\gamma - 1)/(\gamma + 1)] + [2/(\gamma + 1)](1/M^2 \sin^2\beta)\}^{-1}$
G = function to be optimized
g = $\{[2\gamma/(\gamma + 1)]M^2 \sin^2\beta - [(\gamma - 1)/(\gamma + 1)]\}^{-1}$
M = Mach number
m = number of constraints
n = number of shocks
P = pressure
PR = inlet compression ratio
PT = inlet total pressure recovery
P_T = total pressure
x = $1 + [(\gamma - 1)/2]M^2$
y = $1 + [(\gamma - 1)/2]M^2 \sin^2\beta$
β = shock angle
θ = shock turning angle
λ = Lagrange multiplier
ψ = constraint relation

Introduction

IN 1944, Oswatitsch published one of the classical papers of high-speed aerodynamics. In this article (translated into English as Ref. 1), Oswatitsch set out to determine which combination of $n - 1$ oblique shocks and one terminating normal shock reduced a supersonic flow to subsonic with the maximum total pressure recovery. With some intuitive manipulation of gasdynamic relations and the use of Lagrange multipliers, he proved that total pressure recovery is a maximum when all of the oblique shocks have equal strength. Based on the results of this work, the optimal shock strengths, total pressure recoveries, and turning angles for two-dimensional supersonic inlets with up to four shocks were plotted in Ref. 2 for Mach numbers ranging between 1 and 5.

Oswatitsch's analysis is applicable to two-dimensional supersonic inlets for gas-turbine engines. Scramjet engines, however, have somewhat different inlet requirements. A scramjet inlet is usually required to generate a specified pressure rise at a given Mach number, and to supply supersonic flow to a combustor that is typically aligned with the flow entering the inlet. A schematic of a two-dimensional scramjet inlet is shown in Fig. 1. In this inlet, the flow is first compressed and turned by a series of external shocks, then compressed and turned in the opposite direction by a series of internal shocks. This mixed compression results in a supersonic flow entering the combustion chamber at or above the required static pressure. The relative amount of external and internal compression used in a particular inlet is dependent on the relative angle of the entrance and exit flow and inlet starting requirements. Two-dimensional scramjet inlets are optimized in the current work to produce maximum inviscid total pressure recovery by the method of Lagrange multipliers. It is worth noting that this method requires trivial amounts of computation time when compared with more general optimization techniques based on gradient methods or genetic algorithms.

Optimization of a Scramjet Inlet with $n - 1$ External Shocks and One Internal Shock

The constrained optimization problem represented by a scramjet inlet with $n - 1$ external shocks and one internal shock can be solved using the method of Lagrange multipliers. As a general statement of this method,[3] the extreme values of a function $G(x_1, x_2, \ldots, x_l)$, whose l variables are subjected to m constraining relations $\psi_i(x_1, x_2, \ldots, x_l) = 0$ $(i = 1, 2, \ldots, m)$, can be found from the solution of the l equations:

$$\frac{\partial F}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, l$$

and the m constraining relations, where

$$F = G + \sum_{i=1}^{m} \lambda_i \psi_i$$

The parameters λ_i are the so-called Lagrange multipliers. Essentially, this method reduces the determination of extremum values of a function with l unknowns and subject to m constraints, to the solution of a set of $l + m$ equations.

The function to be maximized in the current work is the total pressure recovery of the inlet, PT. The constraints placed upon the problem are

1) Shocks obey the Rankine–Hugoniot relations.
2) Flow exits the inlet parallel with the incoming flow.
3) The inlet compression ratio, PR, is such that the ratio of the freestream dynamic pressure and the inlet exit pressure is unity.

This problem is formulated in the current work in a similar fashion to the supersonic inlet problem of Ref. 1. However, in contrast to the supersonic inlet problem, it does not have an easily recognizable general solution.

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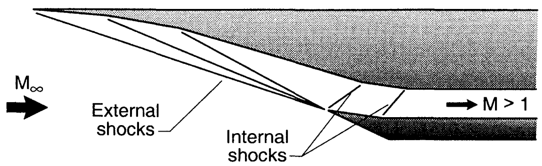

Fig. 1 Schematic of a two-dimensional scramjet inlet.

Figure 2 shows a schematic of the shock-wave system used for the analysis. The $n - 1$ external shocks are labeled with subscripts $0, 1, \ldots, n - 2$, and the internal shock is labeled with subscript $n - 1$. The total pressure ratio across a single shock is given by

$$\frac{P_{T_{i+1}}}{P_{T_i}} = f_i^{\gamma/(\gamma-1)} g_i^{1/(\gamma-1)} \tag{1}$$

where

$$f_i = \left(\frac{\gamma - 1}{\gamma + 1} + \frac{2}{\gamma + 1}\frac{1}{M_i^2 \sin^2\beta_i}\right)^{-1}$$

$$g_i = \left(\frac{2\gamma}{\gamma + 1} M_i^2 \sin^2\beta_i - \frac{\gamma - 1}{\gamma + 1}\right)^{-1}$$

The total pressure recovery of the inlet is then

$$PT = \prod_{i=0}^{n-1} f_i^{\gamma/(\gamma-1)} g_i^{1/(\gamma-1)} \tag{2}$$

For numerical expediency, the natural logs of both sides of Eq. (2) are taken, and we seek to maximize

$$G = (\gamma - 1)\ell n(PT) = \sum_{i=0}^{n-1} (\gamma\, \ell n f_i + \ell n\, g_i) \tag{3}$$

The $2n$ unknowns in the problem are $M_1, M_2, \ldots, M_n$ and $\beta_0, \beta_1, \ldots, \beta_{n-1}$. To simplify the problem, new unknowns are introduced, namely,

$$x_i = 1 + [(\gamma - 1)/2]M_i^2 \tag{4}$$

$$y_i = 1 + [(\gamma - 1)/2]M_i^2 \sin^2\beta_i \tag{5}$$

Shock turning angle, θ_i, as well as f_i and g_i can be expressed in terms of x_i and y_i as

$$\theta_i = \tan^{-1}\left[\sqrt{\frac{x_i - y_i}{y_i - 1}}\, \frac{2y_i - (\gamma + 1)}{(\gamma + 1)x_i - 2y_i}\right] \tag{6}$$

$$f_i = \frac{\gamma + 1}{\gamma - 1}\frac{y_i - 1}{y_i} \tag{7}$$

$$g_i = \frac{\gamma - 1}{\gamma + 1}\frac{1}{[4\gamma/(\gamma + 1)^2]y_i - 1} \tag{8}$$

The constraint that all n shocks must satisfy the Rankine–Hugoniot relations is conveniently expressed as

$$\psi_i = x_{i+1} - x_i f_i g_i = 0, \qquad i = 0, 1, \ldots, n - 1 \tag{9a}$$

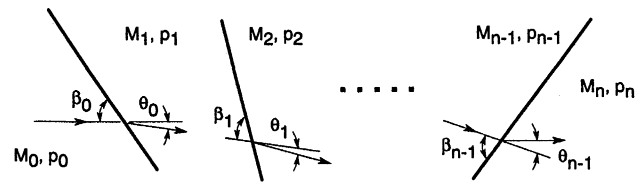

Fig. 2 Shock-wave system used for the analysis.

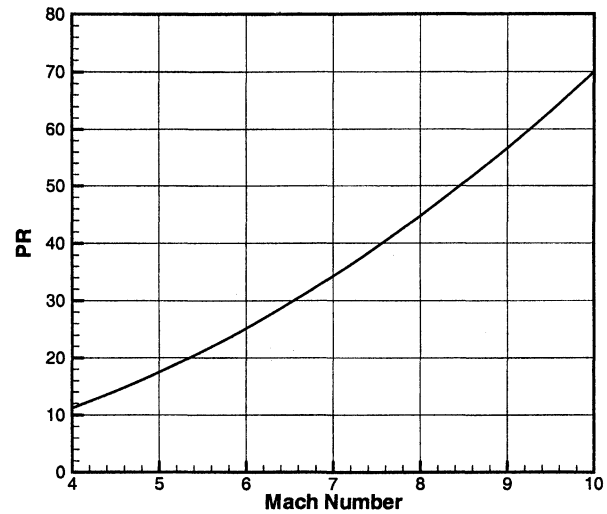

Fig. 3 Required inlet compression ratio.

The flow turning constraint is simply

$$\psi_n = \theta_{n-1} - \sum_{i=0}^{n-2} \theta_i = 0 \tag{9b}$$

Finally, the compression ratio constraint is expressed as

$$\psi_{n+1} = \sum_{i=0}^{n-1} \ell n\, g_i + \ell n(PR) = 0 \tag{9c}$$

Equations (9a–9c) represent the $m = n + 2$ constraints placed on the $2n$ unknowns. The function F is

$$F = G + \sum_{i=0}^{n+1} \lambda_i \psi_i$$

$$F = \sum_{i=0}^{n-1} (\gamma\, \ell n f_i + \ell n\, g_i) + \sum_{i=0}^{n-1} \lambda_i (x_{i+1} - x_i f_i g_i) + \lambda_n \left(\theta_{n-1} - \sum_{i=0}^{n-2} \theta_i\right) + \lambda_{n+1}\left[\sum_{i=0}^{n-1} \ell n\, g_i + \ell n(PR)\right] \tag{10}$$

Equating the partial derivatives of F with respect to $x_1, x_2, \ldots, x_n$ to zero gives the n equations:

$$\lambda_{i-1} - \lambda_i f_i g_i - \lambda_n \frac{d\theta_i}{dx_i} = 0, \qquad i = 1, 2, \ldots, n - 2$$

$$\lambda_{n-2} - \lambda_{n-1} f_{n-1} g_{n-1} + \lambda_n \frac{d\theta_{n-1}}{dx_{n-1}} = 0 \tag{11a}$$

$$\lambda_{n-1} = 0$$

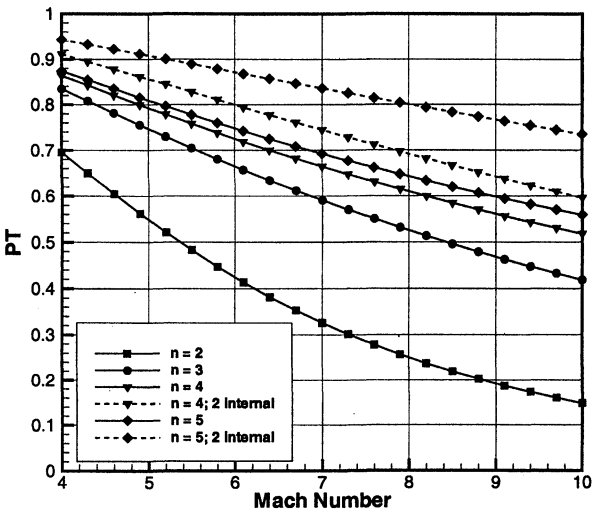

Fig. 4 Maximum total pressure recovery for two-dimensional scramjet inlets with up to five shocks.

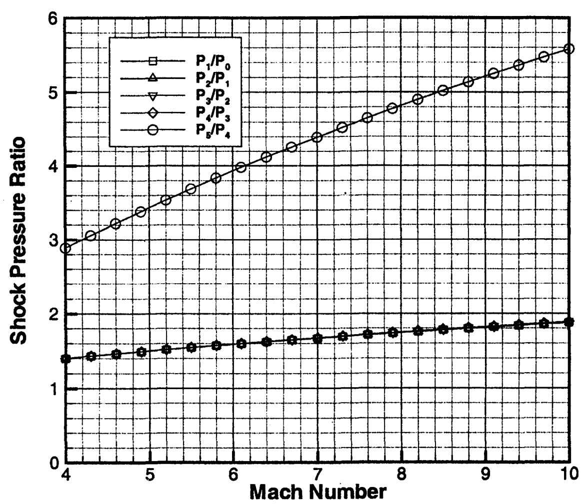

Fig. 5 Shock pressure ratio's for an optimal two-dimensional scramjet inlet with five shocks (one internal shock).

Equating the partial derivatives of F with respect to y_0, y_1, $\dots$, y_{n-1} to zero, gives the n equations:

$$\frac{\gamma}{f_i}\frac{\mathrm{d}f_i}{\mathrm{d}y_i} + \frac{(1+\lambda_{n+1})}{g_i}\frac{\mathrm{d}g_i}{\mathrm{d}y_i} - \lambda_i x_i\left(f_i\frac{\mathrm{d}g_i}{\mathrm{d}y_i} + g_i\frac{\mathrm{d}f_i}{\mathrm{d}y_i}\right) - \lambda_n\frac{\mathrm{d}\theta_i}{\mathrm{d}y_i} = 0, \qquad i = 0, 1, \dots, n-2$$

$$\frac{\gamma}{f_{n-1}}\frac{\mathrm{d}f_{n-1}}{\mathrm{d}y_{n-1}} + \frac{(1+\lambda_{n+1})}{g_{n-1}} - \lambda_{n-1}x_{n-1} \times \left(f_{n-1}\frac{\mathrm{d}g_{n-1}}{\mathrm{d}y_{n-1}} + g_{n-1}\frac{\mathrm{d}f_{n-1}}{\mathrm{d}y_{n-1}}\right) + \lambda_n\frac{\mathrm{d}\theta_{n-1}}{\mathrm{d}y_i} = 0 \tag{11b}$$

Note that f_i, g_i, θ_i and their derivatives, can all be expressed analytically in terms of x_i and y_i, so that Eqs. (11a) and (11b) are algebraic, not differential equations.

Together with the $m = n + 2$ constraining relations [Eqs. (9a–c)], Eqs. (11a) and (11b) represent $3n + 2$ algebraic equations for the $3n + 2$ unknowns x_1, x_2, $\dots$, x_n, y_0, y_1, $\dots$, y_{n-1}, and λ_0, λ_1, $\dots$, λ_{n+1}. It is at this point that the current analysis proceeds differently from Ref. 1. For a supersonic inlet terminated by a normal shock, mathematical manipulation of equations similar to (11a) and (11b) leads to a set of $n - 1$ identical equations for the $n - 1$ oblique shocks. Oswatitsch[1] therefore recognized that the solution to the optimal supersonic inlet problem involved $n - 1$ identical oblique shocks; i.e., oblique shocks of equal strength. Unfortunately, the solution to the optimal scramjet inlet problem is less obvious. Specifically, the terms involving derivatives of θ_i in Eqs. (11a) and (11b), with respect to both x_i and y_i, make simplification of the equation set difficult. In this instance, therefore, the full set

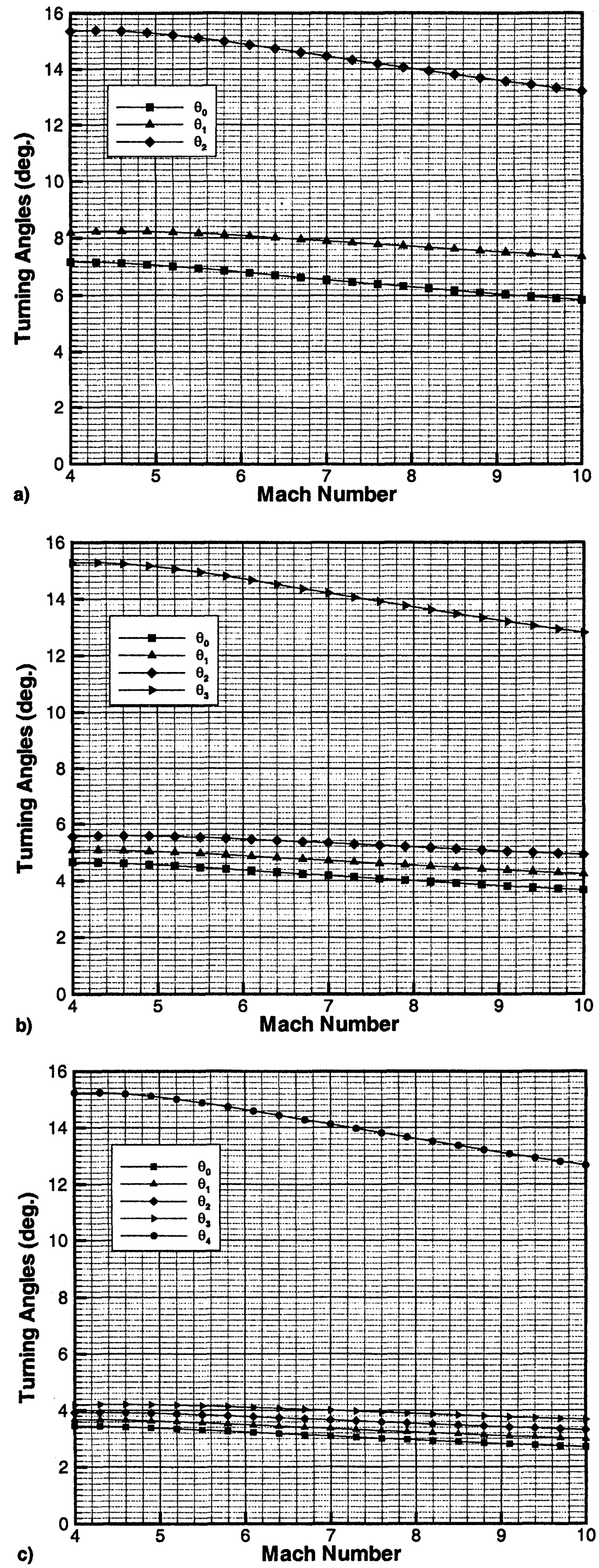

Fig. 6 Optimal turning angles for two-dimensional scramjet inlets with one internal shock: a) $n = 3$, b) $n = 4$, c) $n = 5$ shocks.

of $3n + 2$ equations must be solved for a particular number of shocks using a multi-dimensional secant method such as that described in Ref. 4. Given a reasonable initial guess for the solution, this technique was found to be quite successful at finding a global maxima for scramjet inlet total pressure recovery.

Results and Discussion

The set of $3n + 2$ equations representing the scramjet optimization problem were solved in the current work for $n = 2$, 3, 4, and 5 shocks. The initial solution used to start the iterative solver assumed all external shocks to have equal strength, and all Lagrange multipliers set to zero. Figure 3 shows the inlet compression ratio required to make the exit pressure equal to the freestream dynamic pressure between Mach 4 and 10. Figure 4 shows the optimum total pressure recovery for two-dimensional inlets operating between Mach 4 and 10, which produce the pressure rise indicated in Fig. 3. As expected, the optimum total pressure recovery increases with the number of shocks and decreases with Mach number. Figure 5 shows the pressure ratio across all of the shocks for the case of $n = 5$. It appears that a scramjet inlet with maximum total pressure recovery has external shocks with almost equal strength. However, the internal shock, which is constrained to turn the flow the same magnitude as all the external shocks combined, is considerably stronger. For all cases investigated in the current work, optimum scramjet inlets had external shocks with strengths differing by less than 0.5%. This result is not obvious from an examination of the order of the terms in the equations defining the scramjet inlet optimization problem.

The optimum turning angles for scramjet inlets with $n = 3$, 4, and 5 shocks are plotted in Figs. 6a–6c. Note that the amount of shock turning increases as flow proceeds through the inlet, to keep shock strength almost constant. Also note that the optimal turning angles remain relatively constant over a wide Mach number range. This characteristic can be utilized in the design of inlets required to operate away from the design Mach number.

The analysis described in the previous section may be easily adapted to cases involving two internal shocks. In particular, if the shock labeled $n - 2$ is internal (instead of external), then all terms involving θ_{n-2} or its derivatives in Eqs. (9b), (11a), and (11b) must be changed from positive to negative. No further changes in the analysis are required. The optimal total pressure recoveries for $n = 4$ and 5 shocks (with two of the shocks being internal) are also plotted in Fig. 4. As clearly indicated in Fig. 4, breaking the internal compression into two shocks significantly increases the optimal total pressure recovery of an inlet with four or five shocks. Interestingly, optimal internal shocks have almost the same strength; however, this strength is generally different from the optimal external shock strength because of the imposed flow turning constraint. The choice of one or two internal shocks for a particular inlet depends on the tradeoff between inviscid efficiency, boundary-layer separation control, acceptable internal contraction ratio for inlet starting, and inlet length.

Conclusions

Two-dimensional scramjet inlets with up to five shock waves have been optimized for maximum total pressure recovery using the method of Lagrange multipliers. The constraints placed on the inlets were a compression ratio consistent with a freestream dynamic pressure to exit pressure ratio equal to unity, one internal shock and exit flow parallel with incoming flow. Results showed that optimal scramjet inlets have internal shocks of almost equal strength. Optimal total pressure recoveries and turning angles are plotted for inlets operating between Mach 4 and 10, indicating that optimal turning angles remain relatively constant over a wide Mach number range. For inlets with many shocks, breaking the internal compression into two shocks can significantly improve the optimal inlet total pressure recovery. The method of Lagrange multipliers is an efficient method for optimizing problems with analytically differentiable objective functions, as it requires trivial amounts of computational resources compared with more general optimization techniques.

References

[1]Oswatitsch, K., "Pressure Recovery for Missiles with Reaction Propulsion at High Supersonic Speeds (the Efficiency of Shock Diffusers)," NACA TM 1140 (translation), 1947.

[2]Hermann, R., *Supersonic Inlet Diffusers and Introduction to Internal Aerodynamics*, Honeywell Regulator Co., Minneapolis, MN, 1956, p. 171, Chap. 5.

[3]Sokolnikoff, I., *Advanced Calculus*, McGraw–Hill, New York, 1939, p. 332.

[4]Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P., *Numerical Recipes in Fortran*, 2nd ed., Cambridge Univ. Press, New York, 1992, p. 382, Chap. 9.